

DISORDER AND WETTING TRANSITION: THE PINNED HARMONIC CRYSTAL IN DIMENSION THREE OR LARGER

GIAMBATTISTA GIACOMIN AND HUBERT LACONIN

ABSTRACT. We consider the Lattice Gaussian free field in $d + 1$ dimensions, $d = 3$ or larger, on a large box (linear size N) with boundary conditions zero. On this field two potentials are acting: one, that models the presence of a wall, penalizes the field when it enters the lower half space and one, the *pinning potential*, that rewards visits to the proximity of the wall. The wall can be soft, i.e. the field has a finite penalty to enter the lower half plane, or hard when the penalty is infinite. In general the pinning potential is disordered and it gives on average a reward $h \in \mathbb{R}$ (a negative reward is a penalty): the energetic contribution when the field at site x visits the pinning region is $\beta\omega_x + h$, $\{\omega_x\}_{x \in \mathbb{Z}^d}$ are IID centered and exponentially integrable random variables of unit variance and $\beta \geq 0$. In [3] it is shown that, when $\beta = 0$ (that is, in the non disordered model), a delocalization-localization transition happens at $h = 0$, in particular the free energy of the system is zero for $h \leq 0$ and positive for $h > 0$. We show that, for $\beta \neq 0$, the transition happens at $h = h_c(\beta) := -\log \mathbb{E} \exp(\beta\omega_x)$ and we find the precise asymptotic behavior of the logarithm of the free energy density of the system when $h \searrow h_c(\beta)$. In particular, we show that the transition is of infinite order in the sense that the free energy is smaller than any power of $h - h_c(\beta)$ in the neighborhood of the critical point and that disorder does not modify at all the nature of the transition. We also provide results on the behavior of the paths of the random field in the limit $N \rightarrow \infty$.

2010 *Mathematics Subject Classification:* 60K35, 60K37, 82B27, 82B44

Keywords: Lattice Gaussian Free Field, Disordered Pinning Model, Localization Transition, Critical Behavior, Disorder Irrelevance

1. INTRODUCTION

The object of this paper is the nature of the wetting transition for a d -dimensional harmonic crystal interacting with a substrate and the effect of disorder on this transition.

The harmonic crystal, or lattice Gaussian free field (LGFF), is the basic model for surfaces with Hamiltonian given by the sum of the square of the gradients of the field. Its Gaussian nature makes it, in most of the cases, easier to analyze than other surface fields with gradient potential and conclusions drawn for LGFF are expected to remain valid for a larger class of field.

This is the case for the study of the wetting transition which involves a competition between a repelling potential (possibly infinite) acting on the lower half-space and an attracting one located on a band of finite width above this half-space. What one finds in the literature about this specific problem – the literature on LGFF is very vast since it naturally emerges in a variety of contexts see [16, 17] and references therein – can be resumed as follows:

- In the absence of attracting potential, a wall constraint in the lower half-plane induces a phenomenon of repulsion of entropic origin in dimension $d = 2$ and $d \geq 3$. The surface lies at a distance from the wall which is of order $\log N$ in

dimension two and $\sqrt{\log N}$ in dimension three or larger when N is the size of the system [2, 7, 8, 14] .

- An arbitrary small (positive) *pinning* potential in the intermediate band is sufficient to overcome this entropic repulsion when $d \geq 3$ [3] whereas when $d \geq 2$, the repulsion prevails even in the presence of a small positive potential [4]. So when $d \geq 3$ there is a transition when the potential switches from repulsive to attractive, while the transition happens at some positive value of the pinning potential.

In the present work we analyze the phase transition for $d \geq 3$, with a twofold objective:

- We study the free-energy behavior at the vicinity of the critical point and show that the transition is of infinite order.
- We investigate the effect of disorder on this phase transition and show that quenched and annealed critical points coincide. Moreover we show that the critical behavior is not modified by the disorder.

We also prove that in the localized phase, the distribution of the field in the middle of large box converges when the boundary is sent to infinity to a translation covariant limiting distribution.

These results offer a sharp contrast with those obtained in the absence of half-space repulsion [11, 13] (see also [6] for a first contribution to the subject). In that case, the transition, which is of first order when $d \geq 3$ and of second order when $d = 2$ for the homogeneous case, becomes smoother in the disordered one (order two and infinity respectively). These difference can be interpreted in the light of Harris criterion concerning disorder relevance [12]: for the wetting transition here, the homogeneous model has a smooth transition (the specific heat exponent is negative), and for this reason, disorder should be irrelevant, i.e. it should not change the critical behavior, at least for small perturbation. For the pinning transition studied in [11, 13] the specific heat exponent is positive, so the Harris criterion predicts disorder relevance.

Note that the model is also defined in dimension one: in that case the harmonic crystal is simply a random walk with IID Gaussian increments. In that case the behavior of the model is quite different and very similar to the random walk pinning model (the case where no wall is present) for which an extended literature exists (see [15] for a treatment of the non disordered case and [9, 10] and references therein for one dimensional disordered pinning models).

2. MODEL AND RESULTS

2.1. Wetting models, with and without disorder. Given Λ be a finite subset of \mathbb{Z}^d (Λ is always going to be an hypercube and $d = 3, 4, \dots$), we let $\partial\Lambda$ denote the internal boundary of Λ and $\mathring{\Lambda}$ the set of interior points of Λ , that is (with \sim standing for nearest neighbor)

$$\partial\Lambda = \{x \in \Lambda : \text{there exists } y \notin \Lambda \text{ such that } x \sim y\} \quad \text{and} \quad \mathring{\Lambda} := \Lambda \setminus \partial\Lambda. \quad (2.1)$$

$\mathbf{P}_\Lambda^{\hat{\phi}}$ is the law of the LGFF on Λ (denoted by $\phi = \{\phi_x\}_{x \in \mathbb{Z}^d}$) with boundary conditions $\hat{\phi} \in \mathbb{R}^{\mathbb{Z}^d}$ on $\mathbb{Z}^d \setminus \mathring{\Lambda}$. Explicitly $\phi_x = \hat{\phi}_x$ for $x \notin \mathring{\Lambda}$ and consider $\mathbf{P}_\Lambda^{\hat{\phi}}$ as a probability on $\mathbb{R}^{\mathring{\Lambda}}$

whose density is given by

$$\mathbf{P}_\Lambda^{\hat{\phi}}(d\phi) \propto \exp \left(-\frac{1}{2} \sum_{\substack{(x,y) \in (\Lambda)^2 \\ x \sim y}} \frac{(\phi_x - \phi_y)^2}{2} \right) \prod_{x \in \Lambda} d\phi_x, \quad (2.2)$$

where $\prod_{x \in \Lambda} d\phi_x$ denotes the Lebesgue measure on $\mathbb{R}^{\hat{\Lambda}}$. For the particular case $\hat{\phi} \equiv u$ we write \mathbf{P}_Λ^u . In most of the cases

$$\Lambda = \Lambda_N := \{0, \dots, N\}^d, \quad (2.3)$$

for some (usually large) $N \in \mathbb{N}$, so $\hat{\Lambda}_N := \{1, \dots, N-1\}^d$. We also introduce the notation $\tilde{\Lambda}_N := \{1, \dots, N\}^d$.

Remark 2.1. Of course $\mathbf{P}_\Lambda^{\hat{\phi}}$ is the finite volume LGFF. Much has been written about this field: we stress here that for $d \geq 3$ the $N \rightarrow \infty$ limit, with respect to the product topology, of $\mathbf{P}_{\Lambda_N}^u$ exists and it can be characterized as the Gaussian field with constant expectation u and covariance of ϕ_x and ϕ_y equal to the expected time spent in y by a simple symmetric random walk issued from x (for more on this very well known issue we refer to [11, Sec. 2.9] and references therein). In particular, the variance of ϕ_x in the infinite volume limit does not depend on x and we denote it by σ_d^2 . Moreover the random walk representation holds also in finite volume – the walk is killed at the boundary – and this directly implies that the variance of ϕ_x grows as the region considered grows in the sense of set inclusion.

Given $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ a family of IID square integrable centered random variables (of law \mathbb{P}) with unit variance, we set, for all $\beta \in \mathbb{R}$

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta \omega_x}]. \quad (2.4)$$

We call $I_{\mathbb{P}}$ the interval where $\lambda(\beta)$ is finite and assume that it contains a neighborhood of the origin. The two families of random variables, ω with law \mathbb{P} and the LGFF ϕ with law $\mathbf{P}_\Lambda^{\hat{\phi}}$, are realized on a common probability space and they are independent.

For $x \in \mathbb{Z}^d$ set $\delta_x := \mathbf{1}_{[0,1]}(\phi(x))$ and $\rho_x := \mathbf{1}_{(-\infty,0)}(\phi(x))$. For $\beta \in I_{\mathbb{P}}$, $h \in \mathbb{R}$ and $K \in \mathbb{R} \cup \{+\infty\}$ (but in the main results $K \in (0, \infty]$), we define a modified measure $\mathbf{P}_{N,h,K}^{\beta,\omega,\hat{\phi}}$ via

$$\frac{d\mathbf{P}_{N,h,K}^{\beta,\omega,\hat{\phi}}(\phi)}{d\mathbf{P}_N^{\hat{\phi}}} = \frac{1}{Z_{N,h,K}^{\beta,\omega,\hat{\phi}}} \exp \left(\sum_{x \in \tilde{\Lambda}_N} ((\beta \omega_x - \lambda(\beta) + h)\delta_x - K\rho_x) \right). \quad (2.5)$$

where

$$Z_{N,h,K}^{\beta,\omega,\hat{\phi}} := \mathbf{E}_N^{\hat{\phi}} \left[\exp \left(\sum_{x \in \tilde{\Lambda}_N} ((\beta \omega_x - \lambda(\beta) + h)\delta_x - K\rho_x) \right) \right]. \quad (2.6)$$

In the homogeneous case, $\beta = 0$, we just drop from the superscripts β and ω .

2.2. Main results. We introduce the free energy (density) for every $K \in (-\infty, \infty]$, every $\beta \geq 0$ such that $\lambda(\beta) < \infty$ and every $h \in \mathbb{R}$ as

$$F_K(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N^d} \mathbb{E} \log Z_{N,h,K}^{\beta, \omega, 0}. \quad (2.7)$$

Theorem A.1 ensures that this limit exists, also as an almost sure limit if we drop the expectation with respect to the disorder. We note that, from the free energy viewpoint there is no point in paying attention to summing over $\tilde{\Lambda}_N$ in the energy term $\left(\sum_{x \in \tilde{\Lambda}_N} \dots\right)$ defining the partition function: Λ_N or $\tilde{\Lambda}_N$ give the same free energy. Even more, the measure $\mathbf{P}_{N,h}^{\beta, \omega, \phi}$ does not see these energy changes at the boundary. Where the choice of $\tilde{\Lambda}_N$ enters the game in a non negligible way in relation to the super-additive property: at this stage this important issue is just technical (see Appendix A).

It is very well known that for pinning models the observation that for every $K \in [0, \infty]$ (and every $\beta \in I_{\mathbb{P}}$ and every h)

$$F_K(\beta, h) \geq 0. \quad (2.8)$$

This follows simply from the fact that $-\log \mathbf{P}_N^0(\phi_x > 1 \text{ for every } x \in \tilde{\Lambda}_N) = o(N^d)$ [14]. The bound (2.8) combined with the fact that the convex function $F_K(\beta, \cdot)$ is non decreasing, tells us that there exists $h_{c,K}(\beta)$ (at this stage we drop the dependence on K for conciseness), a priori in $[-\infty, \infty]$, such that $F_K(\beta, h) > 0$ if and only if $h > h_c(\beta)$. Elementary arguments directly yield that $h_c(\beta) \in [h_c(0), h_c(0) + \lambda(\beta)]$ and that $h_c(0) \in [0, \infty)$: $h_c(\beta) \geq h_c(0)$ is just a consequence of the *annealed bound* (Jensen inequality)

$$F(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\log Z_{N,h,K}^{\beta, \omega, h} \right] \leq \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[Z_{N,h,K}^{\beta, \omega, h} \right] = F(\beta, 0). \quad (2.9)$$

The bound $h_c(\beta) \leq h_c(0) + \lambda(\beta)$ follows by convexity of $F(\cdot, h)$, using $\partial_\beta F(0, h) = 0$ (see e.g. [9, p. 23]). Finally $h_c(0) \geq 0$ is just direct consequence of $Z_{N,h,K}^0 \leq 1$ if $h \geq 0$ and $h_c(0) \leq -\log C := -\log \mathbf{P}(\mathcal{N} \in [0, 2d])$ follows from $\mathbf{P}_N^0(\phi_x \in [0, 1] \text{ for every } x \in \tilde{\Lambda}_N) \geq C^{-N^d}$, which one derives by an easy nearest neighbors conditioning argument.

Theorem 2.2. *For every $K \in (0, \infty]$, every $\beta \in I_{\mathbb{P}}$ we have*

$$h_c(\beta) = 0. \quad (2.10)$$

Furthermore if β belongs to the interior of $I_{\mathbb{P}}$ we have that for $h \searrow 0$

$$F_K(\beta, h) = \exp \left(\left(-\frac{\sigma_d^2}{2} + o(1) \right) \left(\log \frac{1}{h} \right)^2 \right). \quad (2.11)$$

The proof of Theorem 2.2 is in Section 3 (Proposition 3.1: case $\beta = 0$ and upper bound estimate for $\beta > 0$) and in Section 4 (Proposition 4.1: lower bound estimate when $\beta > 0$).

It is worth pointing out that also in the case $K = 0$, treated in [11], we have that $h_c(\beta) = 0$, but (2.11) does not hold! In fact in [11] it is shown that the critical behavior in that case has a power law behavior. More importantly, in [11] it is shown that in the $K = 0$ case disorder is relevant, i.e. it changes the critical behavior (for any $\beta > 0$) – while (2.11) shows disorder irrelevance (more on this just below). A less important remark is that the case $K < 0$ can be mapped to the case $K > 0$: the upper half space is in this case penalized and it is an easy matter to work out the correspondences.

Moreover Theorem 2.2 directly generalizes to the case in which δ_x is defined by $b\mathbf{1}_{[0,a]}(\phi_x)$, with a and $b > 0$: (2.11) should simply be replaced by

$$F_K(\beta, h) = \exp \left(\left(-\frac{\sigma_d^2}{2a^2} + o(1) \right) \left(\log \frac{1}{h} \right)^2 \right). \quad (2.12)$$

The novel content of Theorem 2.2 is twofold

- (1) It improves substantially what was known in the literature, and notably the results in [3], where the case $\beta = 0$ has been considered for a pinning potential of the form $b\mathbf{1}_{[0,a]}(\cdot)$, precisely the one addressed by (2.12). In our set-up the potential is rather $h b\mathbf{1}_{[0,a]}(\cdot)$, so we can set $b = 1$ and $h(> 0)$ plays the role of b . The result in [3] can be restated as $h_c(0) = 0$ with a lower bound on the free energy that has not been made explicit by the authors. However, a close look at their computation gives the lower bound

$$F(0, h) \geq \exp(-c_\alpha h^{-\alpha}), \quad (2.13)$$

where $\alpha > 2$ is a constant which depends on c_1 in [3, Proposition 2]. Hence (2.11) improves considerably this bound and provides a matching lower bound. In [3] also a singular limit of the model has been considered: we pick up this model at the end of Section 2.3.

- (2) Our result also covers the disordered case and shows a strong form of disorder irrelevance, in agreement with the Harris criterion, see the review of the literature on this issue in [10, Sec. 5]. In the Harris criterion perspective, Theorem 2.2 finds a parallel in the results on loop exponent one renewal pinning models [1].

It is rather straightforward to extract from the convexity of the free energy that the system has a positive density of contacts if and only if $h > h_c(\beta)$ and therefore $h_c(\beta)$ is the critical point for a localization transition. One can also get a Large Deviation type estimate on the number of contacts in a large volume in the localized regime, like it is done in [3] for the $\beta = 0$ case. And it is precisely in [3] that obtaining pointwise bounds on the field is cited as an open problem. Here we present a result in this direction. For this we choose to work, only for the next statement (and its proof, see Sec. 5), with $\Lambda_N := \{-N, \dots, N\}^d$ (and $\tilde{\Lambda}_N := \{-N+1, \dots, N\}^d$ in (2.5)). The measures we consider are all viewed either as elements of the set of probability measures on $[0, \infty)^{\mathbb{Z}^d}$ or $[0, \infty)^{\mathbb{Z}^d}$, both equipped with the product topology and $[0, \infty]$ is equipped with the usual compactified topology. We use Θ_x for the translation operator, both for ϕ , that is $(\Theta_x \phi)_y = \phi_{x+y}$, and for ω .

Theorem 2.3. *Let us choose $\beta \in I_{\mathbb{P}}$.*

- (1) *If $h > 0$ the sequence of quenched averaged probabilities $\{\mathbb{E}\mathbf{P}_{N,h,\infty}^{\omega,\beta,0}\}_{N=1,2,\dots}$, probabilities on $[0, \infty)^{\mathbb{Z}^d}$, converges to a translation invariant limit. Moreover for \mathbb{P} -almost every ω the sequence $\{\mathbf{P}_{N,h,\infty}^{\omega,\beta,0}\}_{N=1,2,\dots}$, probabilities on $[0, \infty)^{\mathbb{Z}^d}$, converges to a translation covariant limit $\mathbf{P}_{\infty,h,\infty}^{\omega,\beta,0}$, that is $\mathbf{E}_{\infty,h,\infty}^{\omega,\beta,0}[f(\Theta_x \phi)] = \mathbf{E}_{\infty,h,\infty}^{\Theta_x \omega,\beta,0}[f(\phi)]$ for every bounded local f .*
- (2) *If $h \leq 0$ the sequence of quenched averaged and quenched measures, both probabilities on $[0, \infty)^{\mathbb{Z}^d}$, converge to the probability concentrated on the singleton $\{\infty\}^{\mathbb{Z}^d}$.*

Remark 2.4. *Let us observe here that the proof of Theorem 2.3 does not rely much on the assumption $d \geq 3$. When $d = 2$, a non-trivial covariant limit exists when $h > h_c(\beta)$ (as a consequence of [4] and of the annealed bound (2.9) $h_c(\beta) \geq h_c(0) > 0$ for all β) while*

for $h < h_c(\beta)$, the limit is concentrated on $\{\infty\}^{\mathbb{Z}^2}$ (the proof is identical). We cannot conclude in the case $h = h_c(\beta)$ since it is not known whether the localization transition is of first order or higher. In dimension one, results of the same type have been known for a long time (see [10, Section 7.3])

The proof of Theorem 2.3 is in Section 5.

2.3. Outline of the proof of Theorem 2.2. We will first treat, in Section 3, the homogeneous, or pure, model (i.e., $\beta = 0$). This both for presenting the easier case first and because $F_K(\beta, h) \leq F_K(0, h)$ so, in Section 4 dedicated to $\beta > 0$, we just need to provide a lower bound on $F_K(\beta, h)$.

The key point behind Theorem 2.2 is that a one site strategy turns out to be sufficient. Roughly the idea is the following: We can imagine that close to criticality, the field has very few pinned sites (note that this would not be the case if the transition were of first order, but as we are still at the stage of making guesses and it does not seem unreasonable to think that the transition is smooth). Therefore, also helped by the (entropic) repulsion effect of the (soft, $K < \infty$, or hard, $K = \infty$) wall and by the massless character of the field, the field is expected to be at a typical height $u \gg 1$, hence quite far from the levels that contribute to the energy. So the energy contributions are due to rare spikes downwards. The various sharp results on entropic repulsion for LGFF in $d \geq 3$, notably [2, 7, 8], support this intuition and also the fact that large excursions of the LGFF in $d \geq 3$ are essentially just isolated spikes (see [5] for a convergence result of these spikes to a Poisson process): for example, the expectation of $\max_{x \in \Lambda_N} \phi_x$, where ϕ the infinite volume centered LGFF in $d \geq 3$, is, to leading order as $N \rightarrow \infty$, the same as if ϕ were a collection of IID $\mathcal{N}(0, \sigma_d^2)$ random variables (recall that σ_d^2 is the variance of the one dimensional marginal of the infinite volume LGFF). So let us imagine that the field is repelled for h small to a height u very large and that we can look at the contribution of each variable like if they were independent. So we are reduced to the computation of the contribution to the partition function of one site in this idealized setup:

$$\begin{aligned} \mathbf{P}(\phi_x > 1) + e^h \mathbf{P}(\phi_x \in [0, 1]) + e^{-K} \mathbf{P}(\phi_x < 0) = \\ 1 + (e^h - 1) \mathbf{P}(\phi_x \leq 1) + (e^{-K} - e^h) \mathbf{P}(\phi_x < 0). \end{aligned} \quad (2.14)$$

Now, we use $\phi_x \sim \mathcal{N}(u, \sigma_d^2)$, so $\phi_x = \sigma_d \mathcal{N} + u$ (\mathcal{N} is a standard Gaussian variable, or $\mathcal{N} \sim \mathcal{N}(0, 1)$) and the standard asymptotic estimate

$$\mathbf{P}(\mathcal{N} > t) \stackrel{t \rightarrow \infty}{\sim} \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right). \quad (2.15)$$

Since h is small and u is large we can approximate (2.14) by

$$\begin{aligned} 1 + \frac{h\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{(u-1)^2}{2\sigma_d^2}\right) - (1 - e^{-K}) \frac{\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma_d^2}\right) = \\ 1 + \frac{\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma_d^2}\right) \left[h \exp\left(\frac{u}{\sigma_d^2} - \frac{1}{2\sigma_d^2}\right) - (1 - e^{-K}) \right]. \end{aligned} \quad (2.16)$$

Up to now we have not said anything about the value of u , but this computation says that a positive contribution to the free energy requires the term in the square brackets to be positive. And for this one needs to choose $u = \sigma_d^2 \log(1/h) + c$ for some positive constant c and choosing a much larger u would strongly penalize the gain, because of the

prefactor $\exp(-u^2/(2\sigma_d^2))$. Note the marginal role played in this computation by K , as long as $K > 0$. This computation is suggesting that the one site contribution is for $h \searrow 0$

$$1 + \exp\left(-\frac{(\sigma_d^2 + o(1)) \left(\log \frac{1}{h}\right)^2}{2}\right), \quad (2.17)$$

and therefore

$$\begin{aligned} F_K(0, h) &= \log\left(1 + \exp\left(-\frac{(\sigma_d^2 + o(1)) \left(\log \frac{1}{h}\right)^2}{2}\right)\right) \\ &= \exp\left(-\frac{(\sigma_d^2 + o(1)) \left(\log \frac{1}{h}\right)^2}{2}\right). \end{aligned} \quad (2.18)$$

This is the main claim of Theorem 2.2 and in order to convert such an argument into a proof we will proceed separately for upper and lower bound. The upper bound is achieved by reducing the estimate to a model on a (very) spaced sub-lattice (via an application of the Hölder inequality): the Markov property of the LGFF at this point can be used to provide enough independence to obtain the bound we are after. For the lower bound we exploit the fact (in Appendix A) that the free energy can be computed by choosing boundary conditions that are sampled from the infinite volume LGFF with an arbitrary average height u : in fact, with such boundary conditions the logarithm of the partition function forms a super-additive sequence, hence we can perform estimates for finite N to estimate from below the $N = \infty$ case, see (A.5). We then proceed by using Jensen's inequality in a way that a priori may seem very rough (we just compute the expectation of the energy term!), but this turns out to be sufficient, thanks to the first step and the wise choice of the boundary mean u .

For what concerns the disordered case, the desired lower bound on $F_K(\beta, h)$ is achieved once again by exploiting super-additivity – we will work with a finite volume size that diverges as h tends to zero – and by choosing the boundary values at average height u . We choose $u = \sigma_d^2 \log(1/h) + c$. It was argued after (2.16) that this should be the value of u that maximizes the energy gain in the pure case and we choose to use the same value for the disordered case because we are aiming at showing disorder irrelevance. The volume size is chosen so that it is improbable to observe two or more contacts and then the estimate is performed on the partition function limited to the trajectories of the field that have at most one contact (and we send also K to infinity, since, by monotonicity in K of the partition function, this is the worst case scenario). On this reduced free energy we perform a second moment argument. A look at the proof shows that the variance term in this computation plays a very marginal role, reinforcing the idea that disorder is very irrelevant in this model. Even more, we are able to apply the second moment method without assuming that the second moment of $e^{\beta\omega_n}$ is finite! This is achieved with an accurate cut-off procedure. The core of the lower bound argument on $F_K(\beta, h)$ also for $\beta \neq 0$ is in any case the one site computation we have just sketched and the argument in Section 4 can be used verbatim (just set $\beta = 0$) to obtain another (somewhat more involved) proof of the lower bound presented in Section 3 for the non disordered case.

We complete this section by pointing out that our arguments yield partial results about the $\beta = 0$ case treated in [3] under the name of δ -pinning model. This corresponds to the $a \searrow 0$ case of the $b\mathbf{1}_{[0,a]}(\cdot)$ potential of (2.12), with $h = 1$, $b = \exp(c_a + J)$ with $J \in \mathbb{R}$ and c_a such that $\lim_{a \searrow 0} a \exp(c_a) = 1$. The limit model has a number of nice features, but the model is not critical at any value of J : delocalization arises only for $J \rightarrow -\infty$. And in fact

our results show that, with such parametrization, for $a > 0$ the critical value is $J_c = -c_a$, with the corresponding critical behavior directly readable from (2.12). Our arguments of proof suggest that the free energy of the δ -pinning model as $J \rightarrow -\infty$ is equal to

$$\exp\left(-\frac{1}{2}(\sigma_d^2 + o(1))\exp(-2J)\right). \quad (2.19)$$

The upper bound part of this conjecture can be proven in a straightforward way by the arguments we use, but our lower bound arguments do not yield the result because they require a super-additive statement like Proposition A.2 for the δ -pinning model. On the other hand, in [3] no explicit bound is given, but, like for (2.13), reconsidering their approach we find a lower bound for the free energy of $\exp(-\exp(-2\alpha J))$, for $J < 0$ large and α a constant larger than 2.

3. THE HOMOGENEOUS CASE

The main result of this section, Proposition 3.1, implies Theorem 2.2 for $\beta = 0$ and provides the upper bound for the case $\beta > 0$.

Proposition 3.1. *For every $K \in (0, \infty]$ in the limit $h \searrow 0$ we have*

$$F_K(0, h) = \exp\left(\left(-\frac{\sigma_d^2}{2} + o(1)\right)\left(\log \frac{1}{h}\right)^2\right). \quad (3.1)$$

Proof. We treat separately the upper and lower bound.

Upper bound. Since the partition function decreases as K increases, for the upper bound it suffices to prove the statement for a $K > 0$. It is also sufficient to consider $N = nL - 1$, with $n, L \in \mathbb{N}$, L even (both sufficiently large, say larger than 3 at this stage, but later on L is chosen fixed but arbitrarily large and n is sent to ∞) and the quantity

$$Z_{n,L} := \mathbf{E}_N^0 \exp\left(\sum_{x \in \{L, L+1, \dots, (n-1)L-1\}^d} (h \delta_x - K \rho_x)\right). \quad (3.2)$$

Note that the sum in the exponential does not range over the whole box $\tilde{\Lambda}_N$, but there are only $O(n^{d-1}L^d)$ terms missing and for this reason for every fixed L we have

$$\lim_{n \rightarrow \infty} \frac{1}{(Ln)^d} \log Z_{n,L} = \lim_{n \rightarrow \infty} \frac{1}{(Ln)^d} \log Z_{nL-1, h, K}^0 = F(0, h). \quad (3.3)$$

We now set for $v \in \{0, \dots, L-1\}^d =: B_L$

$$\Lambda_{L,n}^v := (v + L\mathbb{Z}^d) \cap \{L, L+1, \dots, (n-1)L-1\}^d, \quad (3.4)$$

and the family $\{\Lambda_{L,n}^v\}_{v \in B_L}$ is a partition of $\{L, L+1, \dots, (n-1)L-1\}^d$ and

$$\begin{aligned} Z_{n,L} &= \mathbf{E}_N^0 \left[\exp\left(\sum_{v \in B_L} \sum_{x \in \Lambda_{L,n}^v} (h \delta_x - K \rho_x)\right) \right] \\ &\leq \prod_{v \in B_L} \left(\mathbf{E}_N^0 \left[\exp\left(L^d \sum_{x \in \Lambda_{L,n}^v} (h \delta_x - K \rho_x)\right) \right] \right)^{L^{-d}}, \end{aligned} \quad (3.5)$$

by Hölder inequality.

Next, we condition on $\{\phi_y\}_{y \in \Gamma_{N,L}^v}$ where

$$\Gamma_{N,L}^v := \left\{ y \in \Lambda_N : \text{there exists } x \in \Lambda_{N,L}^v \text{ such that } \max_{i=1}^d |(y-x)_i| = (L/2) \right\}, \quad (3.6)$$

is just a grid that separates the sites $x \in \Lambda_{L,n}^v$. By the Markov property of the LGFF we readily see that $\{\phi_x\}_{x \in \Lambda_{L,n}^v}$ is a family of conditionally independent Gaussian variables. Their (conditional) mean is given by the harmonic extension of $\{\phi_y\}_{y \in \Gamma_{N,L}^g}$ to the full box and their variance is equal to c_L^2 the variance of free field with zero boundary conditions in the center of a box of side length L , so $\lim_{L \rightarrow \infty} c_L = \sigma_d$. Hence we obtain that almost surely

$$\mathbf{E}_N^0 \left[\exp \left(L^d \sum_{x \in \Lambda_{L,n}^v} (h \delta_x - K \rho_x) \right) \middle| y \in \Gamma_{N,L}^g \right] \leq \left(\sup_{u \in \mathbb{R}} \mathbf{E} \exp \left(L^d h \mathbf{1}_{[0,1]}(c_L \mathcal{N} + u) - L^d K \mathbf{1}_{(-\infty,0)}(c_L \mathcal{N} + u) \right) \right)^{|\Lambda_{L,n}^v|}, \quad (3.7)$$

which yields the same bound for the unconditional expectation. With the notation $P(a, b) = \mathbf{P}(\mathcal{N} \in (a, b))$ we have

$$\begin{aligned} \sup_{u \in \mathbb{R}} \mathbf{E} \exp \left(L^d h \mathbf{1}_{[0,1]}(c_L \mathcal{N} + u) - L^d K \mathbf{1}_{(-\infty,0)}(c_L \mathcal{N} + u) \right) &= \\ 1 + \sup_{u \in \mathbb{R}} \left((\exp(L^d h) - 1)P(-u, -u + 1/c_L) - (1 - \exp(-L^d K))P(-\infty, -u) \right) & \\ \leq 1 + \sup_{u \in \mathbb{R}} \left(2L^d h P(-u, -u + 1/c_L) - \frac{1}{2}P(-\infty, -u) \right), & \end{aligned} \quad (3.8)$$

where the last step we have used $h \leq L^{-d}$ and $(1 - \exp(-L^d K)) \geq 1/2$ (so we assume L larger than a suitable constant dependent on K). We note that the argument of the supremum in the last line in (3.8) is larger than zero if and only if

$$\frac{P(-u, -u + 1/c_L)}{P(-\infty, -u)} > \frac{1}{4L^d h}. \quad (3.9)$$

We now observe that the function

$$(-\infty, \infty) \ni u \mapsto \frac{P(-u, -u + a)}{P(-\infty, -u)} \underset{u \rightarrow \infty}{\sim} \exp \left(au - \frac{a^2}{2} \right), \quad (3.10)$$

is smooth and positive. Moreover it goes to zero as $u \rightarrow -\infty$ and to ∞ as u goes to $+\infty$ (we cite as a fact that this function is increasing, but we do not use it in our proof). Therefore for h small the supremum will be achieved for u large: in particular (3.9) implies that we can restrict the supremum to

$$u \geq \frac{1}{2c_L} + c_L \log \left(\frac{1}{5L^d h} \right) = u_0(h, L) \quad (3.11)$$

By using this information we could get a sharp estimate on the supremum, but we will content ourselves with a much simpler estimate which is sufficient for our purposes. In

fact, neglecting the negative term in (3.8), we obtain that given L and $a < c_L^2/2$ for h sufficiently small we have

$$\begin{aligned} \sup_{u \geq u_0} 2L^d h P(-u, -u + 1/c_L) &= 2L^d h P(-u_0, -u_0 + 1/c_L) \\ &\leq \frac{3L^d h}{\sqrt{2\pi}} \exp\left(-\frac{u_0^2}{2}\right) \leq \exp\left(-a(\log(1/h))^2\right). \end{aligned} \quad (3.12)$$

Therefore, going back to (3.7), we have

$$\mathbf{E}_N^0 \left[\exp \left(L^d \sum_{x \in \Lambda_{L,n}^v} (h \delta_x - K \rho_x) \right) \right] \leq \left(1 + \exp \left(-a \left(\log \frac{1}{h} \right)^2 \right) \right)^{|\Lambda_{L,n}^v|}, \quad (3.13)$$

and from (3.2)

$$\begin{aligned} \frac{1}{(nL)^d} \log Z_{n,L} &\leq \frac{|\Lambda_{L,n}^v|}{(nL)^d} \log \left(1 + \exp \left(-a \left(\log \frac{1}{h} \right)^2 \right) \right) \\ &\leq \frac{2}{L^d} \exp \left(-a \left(\log \frac{1}{h} \right)^2 \right), \end{aligned} \quad (3.14)$$

again for h sufficiently small. By recalling that c_L can be chosen arbitrarily close to σ_d we see that the proof of the upper bound is complete.

Lower bound. Also for the lower bound we work with $K \in (0, \infty)$, but this time we follow the K dependence of the bound. By Proposition A.2, precisely (A.5), and Jensen inequality we obtain that for every u and every N

$$\begin{aligned} \mathbb{F}_K(h) &\geq \frac{1}{N^d} \hat{\mathbf{E}}^u \mathbf{E}_N^{\hat{\phi}} \left[\sum_{x \in \tilde{\Lambda}_N} (h \delta_x - K \rho_x) \right] = \mathbf{E}^u [h \delta_x - K \rho_x] \\ &= \mathbf{E} [h \mathbf{1}_{[0,1]} (\sigma_d \mathcal{N} + u) - K \mathbf{1}_{(-\infty, 0)} (\sigma_d \mathcal{N} + u)]. \end{aligned} \quad (3.15)$$

Therefore, with the notation used for the upper bound we have that for every u

$$\mathbb{F}_K(h) \geq h P(-u, -u + 1/\sigma_d) - K P(-\infty, -u). \quad (3.16)$$

We set $u = \sigma_d \log(1/h) + r$, with r to be determined. Therefore for h sufficiently small (how small depends here on r since we require $u \geq C$ for some deterministic C to use the asymptotic statement (2.15))

$$\begin{aligned} \mathbb{F}_K(h) &\geq \frac{1}{u \sqrt{2\pi}} \left(\frac{h}{2} \exp \left(-\frac{1}{2} \left(u - \frac{1}{\sigma_d} \right)^2 \right) - 2K \exp \left(-\frac{1}{2} u^2 \right) \right) \\ &= \frac{2h^{\sigma_d r} \exp(-r^2/2)}{(r + \sigma_d \log(1/h)) \sqrt{2\pi}} \exp \left(-\frac{\sigma_d^2}{2} \left(\log \frac{1}{h} \right)^2 \right) \left(\frac{e^{-1/(2\sigma_d^2) + r/\sigma_d}}{4} - K \right). \end{aligned} \quad (3.17)$$

We now set $r = \frac{1}{2\sigma_d} + \sigma_d \log(4(K+1))$ and we get to the explicit bound

$$\mathbb{F}_K(h) \geq \frac{2h^{\frac{1}{2} + \sigma_d^2 \log(4(K+1))} \exp \left(-\frac{1}{2} \left(\frac{1}{2\sigma_d} + \sigma_d \log(4(K+1)) \right)^2 \right)}{\left(\frac{1}{2\sigma_d} + \sigma_d \log(4(K+1)) + \sigma_d \log(1/h) \right) \sqrt{2\pi}} e^{-\frac{\sigma_d^2}{2} (\log \frac{1}{h})^2}. \quad (3.18)$$

Therefore for every $b > \sigma_d^2/2$ and every $K \in (0, \infty)$ there exists $h_0 > 0$ such that for every $h \in (0, h_0)$

$$F_K(h) \geq \exp \left(-b \left(\log \frac{1}{h} \right)^2 \right). \quad (3.19)$$

This completes the proof of the lower bound, except for the case $K = \infty$.

For the lower bound in the case $K = \infty$ we observe that for K large r becomes large too and (3.18) holds for arbitrary $h_0 > 0$ as $K \rightarrow \infty$ (but in our formulas we want to have $\log(1/h) \geq 0$ so $h_0 = 1$). Now remark that if we choose $K + 1 = \exp((\log(1/h))^{1/2})$ the ratio in right-hand side of (3.18) is bounded below by $\exp(-(\log(1/h))^a)$ for any $a > 3/2$ and h sufficiently small. So for every $b > \sigma_d^2/2$ there exists $h_1 > 0$ such that for every $h \in (0, h_1)$ (3.19) holds. The conclusion is then an immediate consequence of Lemma A.3, because $F_\infty(0, h) \geq F_K(0, h) - \exp(-K)$. This completes the proof of the lower bound and therefore the proof of Proposition 3.1. \square

4. THE DISORDERED CASE

Proposition 4.1. *For every $\beta \in I_{\mathbb{P}}$ we have $h_c(\beta) = 0$. Moreover if β is in the interior of $I_{\mathbb{P}}$, for every $K \in (0, \infty]$ and every $\varepsilon > 0$ there exists h_0 such that*

$$F_K(\beta, h) \geq \exp \left(-(1 + \varepsilon) \frac{\sigma_d^2}{2} \left(\log \frac{1}{h} \right)^2 \right), \quad (4.1)$$

for $h \in (0, h_0)$.

In this section we set

$$\xi_x := e^{\beta \omega_x - \lambda(\beta)}, \quad (4.2)$$

and let ξ denote a variable which has the same distribution as all the ξ_x . Note that $\mathbb{E}\xi = 1$ and that the assumption that β is in the interior of $I_{\mathbb{P}}$ is equivalent to

$$\text{“ There exist } C > 0 \text{ and } \gamma > 1 \text{ such that } \mathbb{P}(\xi \geq t) \leq Ct^{-\gamma} \text{ for every } t \geq 0. \text{”} \quad (4.3)$$

We also use the notation $\rho_x^+ := \mathbf{1}_{(-\infty, 1]}(\phi_x)$. For $\tilde{a} > 1$ we set

$$u := \tilde{a} \sigma_d^2 |\log h| \quad \text{and} \quad N = \exp(|\log h|^{3/2}). \quad (4.4)$$

A basic recurrent quantity in the proof is going to be

$$P(u) := \mathbf{P}^u(\phi_0 \leq 1) \begin{cases} \stackrel{h \searrow 0}{\sim} \frac{\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{(u-1)^2}{2\sigma_d^2}\right), \\ \geq \exp\left(-(1 + \varepsilon) \frac{\tilde{a}^2 \sigma_d^2}{2} |\log h|^2\right), \end{cases} \quad (4.5)$$

where we have used (2.15) and the inequality, which holds for every $\varepsilon > 0$ and h sufficiently small, is directly obtained by inserting the value of u . Moreover for every $c < 1$ we have $\mathbf{P}^u(\phi_0 \in [c, 1]) \sim P(u)$, for $u \rightarrow \infty$. Another relevant estimate is

$$\frac{\mathbf{P}^u(\phi_0 \leq 1)}{\mathbf{P}^u(\phi_0 < 0)} \stackrel{h \searrow 0}{\sim} h^{-\tilde{a}} \exp(-1/(2\sigma_d^2)). \quad (4.6)$$

Here and in the remainder of the proof, we avoid insisting on the fact that we should choose h such that $\exp(|\log h|^{3/2}) \in \mathbb{N}$: obtaining the estimate along this subsequence yields the claim for $h \searrow 0$ by a direct estimate and using that $F_K(\beta, \cdot)$ is non decreasing. Alternatively one can carry along the proof $N = \lfloor \exp(|\log h|^{3/2}) \rfloor$ and deal with the little nuisances that arise.

We introduce also the event

$$G_u := \left\{ \phi \in \mathbb{R}^{\mathbb{Z}^d} : \phi_x > \frac{u}{2} \text{ for } x \in \partial\Lambda_N \right\}. \quad (4.7)$$

The following statement controls the contribution of the bad boundary configurations:

Lemma 4.2. *For every d there exist $C_d > 0$ and $h_0 > 0$ such that for every $K \geq 0$, $\beta \in I_{\mathbb{P}}$ and for $h \in [0, h_0)$ we have*

$$\mathbb{E} \hat{\mathbf{E}}^u \left[\left(\log Z_{N,h,K}^{\beta, \omega, \hat{\phi}} \right) \mathbf{1}_{G_u^c}(\hat{\phi}) \right] \geq -C_d (\lambda(\beta) \vee K) N^{d-1} P(u). \quad (4.8)$$

Proof. By Jensen inequality

$$\begin{aligned} \mathbb{E} \hat{\mathbf{E}}^u \left[\left(\log Z_{N,h,K}^{\beta, \omega, \hat{\phi}} \right) \mathbf{1}_{G_u^c}(\hat{\phi}) \right] &\geq (-\lambda(\beta) + h) \mathbf{E}^u \left[\sum_{x \in \tilde{\Lambda}_N} \delta_x; G_u^c \right] - K \mathbf{E}^u \left[\sum_{x \in \tilde{\Lambda}_N} \rho_x; G_u^c \right] \\ &\geq -(\lambda(\beta) \vee K) \mathbf{E}^u \left[\sum_{x \in \tilde{\Lambda}_N} \rho_x^+; G_u^c \right], \end{aligned} \quad (4.9)$$

with the notation $\mathbf{E}^u[\cdot; F] = \mathbf{E}^u[\cdot \mathbf{1}_F(\phi)]$. By the union bound and by making an elementary splitting for a $C > 0$ we have

$$\begin{aligned} \mathbf{E}^u \left[\sum_{x \in \tilde{\Lambda}_N} \rho_x^+; G_u^c \right] &\leq \sum_{\substack{x \in \tilde{\Lambda}_N \\ y \in \partial\Lambda_N}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2) \\ &\leq \sum_{\substack{x \in \tilde{\Lambda}_N, y \in \partial\Lambda_N: \\ |x-y| \leq C}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2) + \sum_{\substack{x \in \tilde{\Lambda}_N, y \in \partial\Lambda_N: \\ |x-y| > C}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2) \\ &\leq 2dC N^{d-1} P(u) + \sum_{\substack{x \in \tilde{\Lambda}_N, y \in \partial\Lambda_N: \\ |x-y| > C}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2). \end{aligned} \quad (4.10)$$

Now given $\eta > 0$ we have

$$\mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2) \leq \mathbf{P}^u((\phi_x + \eta\phi_y) \leq 1 + u\eta/2). \quad (4.11)$$

Now $\phi_x + \eta\phi_y$ is a Gaussian variable of mean $(1 + \eta)u$. To compute its variance we observe that the covariance between ϕ_x and ϕ_y is given by $G(x, y) = \sigma_d^2 p(x, y)$ with $p(x, y)$ the probability that a simple random walk issued from x hits y : since $p(x, y)$ vanishes when $|x - y|$ becomes large, we choose C so that $p(x, y) \leq 1/8$ when $|x - y| \geq C$. Hence we have for $\eta \leq 1/4$

$$\text{var}(\phi_x + \eta\phi_y) \leq \sigma_d^2(1 + \eta^2 + \eta/4) \leq \sigma_d^2(1 + \eta/2). \quad (4.12)$$

Using this information we have for u sufficiently large

$$\begin{aligned} \mathbf{P}^u((\phi_x + \eta\phi_y) \leq 1 + u\eta/2) &\leq \exp \left(-\frac{(u(1 + \eta/2) - 1)^2}{\sigma_d^2(2 + \eta)} \right) \\ &\leq P(u) \exp(-c(\eta)u^2) \leq N^{-d} P(u), \end{aligned} \quad (4.13)$$

where $c(\eta) = \eta/(8\sigma_d^2)$ and we have used the first line in (4.5) together with the relation between the parameters (4.4).

Hence, going back to (4.10), we see that

$$\mathbf{E}^u \left[\sum_{x \in \tilde{\Lambda}_N} \rho_x^+; G_u^c \right] \leq (2d(C+1)N^{d-1})P(u). \quad (4.14)$$

By plugging this estimate into (4.9) we complete the proof. \square

Proof of Proposition 4.1. We aim at producing a lower bound on $Z_{N,h,K}^{\beta,\omega,\hat{\phi}}$ for good boundary values $\hat{\phi}$ (Lemma 4.2 is going to take care of the bad ones). This will be achieved by a second moment approach: we give first the proof assuming that the second moment of ξ is finite, that is $\lambda(2\beta) < \infty$, or $2\beta \in I_{\mathbb{P}}$. Then we will show how to relax this condition.

The second moment method is not applied directly to the partition function, but to a *reduced* version for which we allow at most one contact in $[0, 1]$ and none in $(-\infty, 0)$. Note in fact that

$$\begin{aligned} Z_{N,h,K}^{\beta,\omega,\hat{\phi}} &\geq Z_{N,h,K}^{\beta,\omega,\hat{\phi}} \left(\left\{ \phi : \sum_{x \in \tilde{\Lambda}_N} \delta_x \in \{0, 1\}, \sum_{x \in \tilde{\Lambda}_N} \rho_x = 0 \right\} \right) \\ &= \mathbf{P}_N^{\hat{\phi}} \left(\phi_x > 1 \text{ for } x \in \tilde{\Lambda}_N \right) \\ &\quad + \sum_{x \in \tilde{\Lambda}_N} e^{\beta\omega_x - \lambda(\beta) + h} \mathbf{P}_N^{\hat{\phi}} \left(\delta_x = 1 \text{ and } \sum_{y \in \tilde{\Lambda}_N \setminus \{x\}} \rho_y^+ = 0 \right) =: Q_{N,h}^{\beta,\omega,\hat{\phi}}, \end{aligned} \quad (4.15)$$

and for conciseness we write $Q_N^{\omega,\hat{\phi}}$ for $Q_{N,h}^{\beta,\omega,\hat{\phi}}$: this is the reduced partition function. Note that the reduced partition function does not contain K and in fact (4.15) holds uniformly in $K \geq 0$.

The first observation on $Q_N^{\omega,\hat{\phi}}$ is that

$$\begin{aligned} Q_N^{\omega,\hat{\phi}} &\geq \mathbf{P}_N^{\hat{\phi}} \left(\phi_x > 1 \text{ for } x \in \tilde{\Lambda}_N \right) \\ &= 1 - \mathbf{P}_N^{\hat{\phi}} \left(\cup_{x \in \tilde{\Lambda}_N} \{\phi_x \leq 1\} \right) \geq 1 - \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}} (\phi_x \leq 1), \end{aligned} \quad (4.16)$$

and a direct estimate shows that, if $\hat{\phi} \in G_u$, we can find $c > 0$ such that

$$\mathbf{P}_N^{\hat{\phi}} (\phi_x \leq 1) \leq \exp(-c(\log N)^2)$$

for every $x \in \tilde{\Lambda}_N$. Hence, for h sufficiently small we have $Q_N^{\omega,\hat{\phi}} \geq 1/2$ and therefore

$$\log Q_N^{\omega,\hat{\phi}} \geq (Q_N^{\omega,\hat{\phi}} - 1) - (Q_N^{\omega,\hat{\phi}} - 1)^2. \quad (4.17)$$

which leads to the bound (uniform in $K \geq 0$)

$$\mathbb{E} \hat{\mathbf{E}}^u \left[\left(\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}} \right) \mathbf{1}_{G_u}(\hat{\phi}) \right] \geq \mathbb{E} \hat{\mathbf{E}}^u \left[\left(Q_N^{\omega,\hat{\phi}} - 1 \right) \mathbf{1}_{G_u}(\hat{\phi}) \right] - \mathbb{E} \hat{\mathbf{E}}^u \left[\left(Q_N^{\omega,\hat{\phi}} - 1 \right)^2 \mathbf{1}_{G_u}(\hat{\phi}) \right], \quad (4.18)$$

on which we will concentrate our attention from here till the end of the proof.

First moment estimates (lower and upper bound). Let us observe that $\mathbb{E}(Q_N^{\omega, \hat{\phi}} - 1)$ is equal to

$$- \mathbf{P}_N^{\hat{\phi}} \left(\bigcup_{x \in \tilde{\Lambda}_N} \{\phi_x \leq 1\} \right) + e^h \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}} \left(\delta_x = 1 \text{ and } \phi_y > 1 \text{ for } y \in \tilde{\Lambda}_N \setminus \{x\} \right), \quad (4.19)$$

and that this quantity, by the union bound, using also $e^h - 1 \geq h$ and the notation $F_x := \{\phi_y > 1 \text{ for } y \in \tilde{\Lambda}_N \setminus \{x\}\}$, can be bounded below by

$$\begin{aligned} & - \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) + \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1) - \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x^c) \\ & \quad + h \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1) - h \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x^c), \end{aligned} \quad (4.20)$$

which we reorder into

$$\begin{aligned} \mathbb{E}(Q_N^{\omega, \hat{\phi}} - 1) & \geq h \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) - (1 + h) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x < 0) \\ & \quad - (1 + h) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x^c). \end{aligned} \quad (4.21)$$

Now we observe that

$$\mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x^c) \leq \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1) \max_{z \in [0, 1]} \sum_{y \in \tilde{\Lambda}_N \setminus \{x\}} \mathbf{P}_N^{\hat{\phi}}(\phi_y \leq 1 \mid \phi_x = z), \quad (4.22)$$

and we use the fact that the probability that a random walk issued from y hits $\partial\Lambda_N$ before visiting x is larger than the probability that a random walk in \mathbb{Z}^d issued from y never hits x , and this latter probability q is positive. Hence the mean of ϕ_y , under $\mathbf{P}_N^{\hat{\phi}}(\cdot \mid \phi_x = z)$, is at least $qu/2$, because $\hat{\phi} \in G_u$, and therefore, since the variance is bounded (by σ_d^2), there exists $c > 0$ such that $\mathbf{P}_N^{\hat{\phi}}(\phi_y \leq 1 \mid \phi_x = z) \leq \exp(-c(\log h)^2)$ and therefore the last term in (4.21) is negligible with respect to the first in the right-hand side of the same formula. Therefore for $\hat{\phi} \in G_u$ and for h sufficiently small we have

$$\mathbb{E}(Q_N^{\omega, \hat{\phi}} - 1) \geq \frac{4}{5}h \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) - (1 + h) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x < 0). \quad (4.23)$$

We will also need an upper bound on $\mathbb{E}(Q_N^{\omega, \hat{\phi}} - 1)$. For this we restart from (4.19) and observe that

$$\begin{aligned} \mathbb{E}(Q_N^{\omega, \hat{\phi}} - 1) & = -\mathbf{P}_N^{\hat{\phi}} \left(\bigcup_{x \in \tilde{\Lambda}_N} \{\phi_x \leq 1\} \right) + e^h \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x) \\ & \leq (e^h - 1) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x) \\ & \leq (e^h - 1) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1). \end{aligned} \quad (4.24)$$

Second moment estimate. Recall that we assume here that $\mathbb{E}[\xi^2] < \infty$. First of all

$$\mathbb{E} \left[\left(Q_N^{\omega, \hat{\phi}} - 1 \right)^2 \right] = \left(\mathbb{E} \left(Q_N^{\omega, \hat{\phi}} - 1 \right) \right)^2 + \text{var}_{\mathbb{P}} \left(Q_N^{\omega, \hat{\phi}} \right). \quad (4.25)$$

The variance term is easily computed and estimated:

$$\begin{aligned} \text{var}_{\mathbb{P}} \left(Q_N^{\omega, \hat{\phi}} \right) &\leq e^{2h} \text{var}(\xi) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1)^2 \\ &\leq e^{2h} \text{var}(\xi) \max_{x' \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_{x'} = 1) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1). \end{aligned} \quad (4.26)$$

For the square of the mean the estimate is already in (4.24). Hence

$$\begin{aligned} \mathbb{E} \left[\left(Q_N^{\omega, \hat{\phi}} - 1 \right)^2 \right] &\leq \left(e^{2h} \text{var}(\xi) + (e^h - 1)^2 N^d \right) \max_{x' \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_{x'} = 1) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1) \\ &\leq 2N^d \max_{x' \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_{x'} = 1) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1) \\ &\leq \exp(-c(\log h)^2) \sum_{x \in \tilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1). \end{aligned} \quad (4.27)$$

In both inequalities we used that h is small (how small depends on $\text{var}(\xi)$: we require $e^{2h} \text{var}(\xi) \leq N^d$ and $(e^h - 1)^2 \leq 1$) and that $N = \exp(|\log h|^{3/2})$, and in the last inequality we used $\hat{\phi} \in G_u$ (which ensures that the mean of ϕ_x under $\mathbf{P}_N^{\hat{\phi}}$ is of order $|\log h|$). Note that the constant c does not depend on ξ . We have insisted on the role of ξ to prepare the generalization to the case in which ξ has unbounded second moment.

Lower bound on $\log Z$. We go back to (4.18): uniformly in $K \geq 0$

$$\begin{aligned} \mathbb{E} \hat{\mathbf{E}}^u \left[\left(\log Z_{N,h,K}^{\beta, \omega, \hat{\phi}} \right) \mathbf{1}_{G_u}(\hat{\phi}) \right] &\geq \frac{4}{5} h \sum_{x \in \tilde{\Lambda}_N} \hat{\mathbf{E}}^u \left[\mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) \mathbf{1}_{G_u}(\hat{\phi}) \right] \\ &\quad - (1+h) |\tilde{\Lambda}_N| \mathbf{P}^u(\phi_0 < 0) - \exp(-c(\log h)^2) |\tilde{\Lambda}_N| \mathbf{P}^u(\delta_0 = 1) \\ &\geq \frac{4}{5} h N^d \mathbf{P}^u(\phi_0 \leq 1) - \frac{4}{5} h \sum_{x \in \tilde{\Lambda}_N} \hat{\mathbf{E}}^u \left[\mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) \mathbf{1}_{G_u^c}(\hat{\phi}) \right] \\ &\quad - (1+h) N^d \mathbf{P}^u(\phi_0 < 0) - \exp(-c(\log h)^2) N^d \mathbf{P}^u(\delta_0 = 1) \\ &\geq \frac{4}{5} h N^d P(u) - \frac{4}{5} h \mathbf{E}^u \left[\sum_{x \in \tilde{\Lambda}_N} \rho_x^+; G_u^c \right] - h^b N^d P(u), \end{aligned} \quad (4.28)$$

with $b \in (1, \tilde{a})$ (recall that $\tilde{a} > 1$), and h sufficiently small. In the last inequality we have controlled from below the two terms in the line before the last one by $-h^b N^d P(u)$: this is because (4.6) tells us $\mathbf{P}^u(\phi_0 < 0) = O(h^{\tilde{a}}) \mathbf{P}^u(\phi_0 \leq 1)$, so the first term in the line before the last one in (4.28) is much larger than the second and all this line is controlled as we claimed. The second term in the last line of (4.28) has been already treated in (4.14) and

we readily see that it is negligible with respect to the first. Therefore we get to

$$\mathbb{E}\hat{\mathbf{E}}^u \left[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}}; G_u \right] \geq \frac{2}{3}hN^d P(u), \quad (4.29)$$

for h sufficiently small and by Lemma 4.2 for every $K \geq 0$

$$\mathbb{E}\hat{\mathbf{E}}^u \left[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}} \right] \geq \left(\frac{2}{3}h - C_d h^2 (\lambda(\beta) \vee K) \right) N^d P(u), \quad (4.30)$$

so the proof of Proposition 4.1, assuming $\mathbb{E}[\xi^2] < \infty$ is complete, for every $K > 0$. For the case $K = \infty$ we apply Lemma A.6 – recall also (A.7) – with $K = h^{-1/2}$ and (4.30).

Relaxing the assumption $E[\xi^2] < \infty$. Let us assume now only that $\beta \in I_{\mathbb{P}}$, that is (4.3). We then replace ξ by $\xi_H := \min(\xi, H)$ in the partition function. Of course we have $E[\xi_H] < 1$. However if we rescale u accordingly all the computations of the above remain valid if one chooses

$$h = -\log \mathbb{E}[\xi_H] + s, \quad (4.31)$$

with $s > 0$ and choose $N = \exp(|\log s|^{3/2})$. In this setup s plays the role of h . We obtain in particular that there exists $s_0 := s_0(H, \varepsilon)$ such that for $s < s_0$

$$\mathbb{F}_K(\beta, -\log \mathbb{E}[\xi_H] + s) \geq \exp \left(-(1 + \varepsilon) \frac{\sigma_d^2}{2} \left(\log \frac{1}{s} \right)^2 \right), \quad (4.32)$$

and from this we extract that $h_c(\beta) \leq -\log \mathbb{E}[\xi_H]$, which, sending $H \rightarrow \infty$, yields $h_c(\beta) = 0$.

To obtain a lower bound on the free energy we assume that β is in the interior of $I_{\mathbb{P}}$ and we make explicit the estimate by carefully tracking the H dependence in the lower bound proof. Of course the first moment estimates do not depend on the value of H , and browsing the part involving the second moment, we can check that the variance of ξ only intervenes in (4.26)-(4.27). It suffices that

$$\exp(2h) \text{var}(\xi_H) \leq N^d, \quad (4.33)$$

and since trivially $\text{var}(\xi_H) \leq H^2$, $N \geq H$ suffices. Recalling (4.4), if $s \leq \exp(-(\log H)^{2/3})$ and if H is sufficiently large, how large depends on ε , (4.32) holds. On the other hand we have from (4.3)

$$-\log \mathbb{E}(\xi_H) = -\log \left(1 - \int_H^\infty \mathbb{P}(\xi > t) dt \right) \leq \frac{C}{\gamma - 1} H^{-(\gamma-1)}. \quad (4.34)$$

Hence for small h we can fix $s = h/2$ and $H = \exp(|\log h|^{3/2})$ and we deduce from (4.32) that

$$\mathbb{F}_K(\beta, h) \geq \mathbb{F}_K(\beta, -\log \mathbb{E}[\xi_H] + h/2) \geq \exp \left(-(1 + \varepsilon) \frac{\sigma_d^2}{2} \left(\log \frac{2}{h} \right)^2 \right), \quad (4.35)$$

and the proof of Proposition 4.1 is now complete. \square

Remark 4.3. A look at the previous argument shows that it goes through even weakening a little (4.32), therefore including cases in which $\lim_{t \rightarrow \infty} t^\gamma \mathbb{P}(\xi > t) = \infty$, for any $\gamma > 1$, but it is equal to zero for $\gamma = 1$. This means that in some cases it works also for β at the boundary of $I_{\mathbb{P}}$. However if the tail decay is too weak (e.g. $\mathbb{P}(\xi \geq t) \geq (t(\log t))^{-1}(\log \log t)^{-2}$), (2.11) does not hold as it can be seen by applying and adapting the upper bound argument present in [11].

5. INFINITE VOLUME LIMIT: PROOF OF THEOREM 2.3

We recall that for Theorem 2.3, we have chosen $\Lambda_N = \{-N, \dots, N\}^d$, and $\tilde{\Lambda}_N$ accordingly. In this section we always assume that $\beta \in I_{\mathbb{P}}$.

The first remark is that $\{\mathbf{P}_{N,h}^{\beta,\omega}\}_{N=1,2,\dots}$ is increasing for the order induced by stochastic domination. Later on we will use also the more general statement

$$\mathbf{P}_{\Lambda,h}^{\beta,\omega} \leq \mathbf{P}_{\Lambda',h}^{\beta,\omega} \quad \text{if } \Lambda \supset \Lambda'. \quad (5.1)$$

where \leq stands here for stochastic domination. Hence we can couple the family of random variables $\phi^N = \{\phi^N\}_{x \in \mathbb{Z}^d}$ with law $\mathbf{P}_{N,h}^{\beta,\omega} := \mathbf{P}_{N,h,\infty}^{\beta,\omega}$ in a way that ϕ^N increases with N . Therefore for every local continuous function $f : [0, \infty]^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ we have that for every ω

$$\lim_{N \rightarrow \infty} \mathbf{E}_{N,h}^{\beta,\omega} [f(\phi)] = \mathbf{E}_h^{\beta,\omega} [f(\phi)], \quad (5.2)$$

and, by the Dominated Convergence Theorem, the same holds by taking the \mathbb{E} expectation on both sides (we are using $\mathbf{E}_h^{\beta,\omega}$ for $\mathbf{E}_{\infty,h}^{\beta,\omega}$).

We are now going to argue that $\mathbf{P}_h^{\beta,\omega}$ satisfies the Markov property. We use the notation \mathcal{F}_A for the σ -algebra generated by $\phi_A := \{\phi_x\}_{x \in A}$, $A \subset \mathbb{Z}^d$. The aim is showing that for every finite subset Γ of \mathbb{Z}^d and for every local bounded continuous $g : [0, \infty]^\Gamma \rightarrow \mathbb{R}$ – in particular, the limit of $g(\phi_\Gamma)$, when $\phi_x \rightarrow \infty$ for every $x \in \Gamma$, exists and we call it $g(\infty)$ – for all $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ we have that $\mathbf{P}_h^{\beta,\omega}(\mathrm{d}\tilde{\phi})$ -a.s.

$$\begin{aligned} \mathbf{E}_h^{\beta,\omega} [g(\phi_\Gamma) \mid \mathcal{F}_{\Gamma^c}] (\tilde{\phi}) = \\ \begin{cases} \frac{1}{Z_{\Gamma,h}^{\beta,\omega,\phi,+}} \mathbf{E}_\Gamma^{\tilde{\phi},+} [\exp(\sum_{x \in \Gamma} (\beta\omega - \lambda(\beta) + h)\delta_x) g(\phi_\Gamma)] & \text{if } \max \phi_{\partial^+ \Gamma} < \infty, \\ g(\infty) & \text{if } \max \phi_{\partial^+ \Gamma} = \infty, \end{cases} \end{aligned} \quad (5.3)$$

where $\partial^+ \Gamma := \{y \in \Gamma^c : \text{there exists } x \in \Gamma \text{ such that } y \sim x\}$ and $\mathbf{P}_\Gamma^{\tilde{\phi},+}$ is the law of a free field ϕ with boundary condition $\tilde{\phi}$ on $\mathbb{Z}^d \setminus \Gamma$, recall (2.2), conditioned to $\{\phi : \phi_x \geq 0 \text{ for every } x \in \Gamma\}$. $Z_{\Gamma,h}^{\beta,\omega,\phi,+}$ is the obvious normalization constant associated to the Boltzmann term that appears in the right-hand side.

Call $G_{g,\Gamma}(\tilde{\phi})$ the right-hand side of (5.3). Two important observations are:

- (1) $G_{g,\Gamma}(\tilde{\phi})$ depends only on $\phi_{\partial^+ \Gamma}$: this is the Markov property. We will then consider $G_{g,\Gamma}(\cdot)$ as a function from $[0, \infty]^{\partial^+ \Gamma}$ to \mathbb{R} .
- (2) Of course $\|G_{g,\Gamma}\|_\infty \leq \|g\|_\infty$ and one directly verifies also the continuity of $G_{g,\Gamma}(\cdot)$.

To prove (5.3) it suffices to show that for every bounded local continuous $f : [0, \infty]^{\Gamma^c} \rightarrow \mathbb{R}$ we have that

$$\mathbf{E}_h^{\beta,\omega} [f(\phi_{\Gamma^c}) g(\phi_\Gamma)] = \mathbf{E}_h^{\beta,\omega} [f(\phi_{\Gamma^c}) G_{g,\Gamma}(\phi)]. \quad (5.4)$$

But, by continuity and boundedness of the integrands, in both sides of (5.4) we can replace $\mathbf{E}_h^{\beta,\omega}[\dots]$ with $\lim_N \mathbf{E}_h^{N,\beta,\omega}[\dots]$ and the finite volume statement is directly verified as soon as N is sufficiently large, since the locality of f implies that $f(\phi) = f(\phi')$ if $\phi_{\Lambda_N} = \phi'_{\Lambda_N}$ for N larger than a finite value that depends on f . So (5.3) holds and the infinite volume field we built satisfies the Markov property.

Next we prove that the quenched measure $\mathbf{P}_h^{\beta,\omega}$ is translationally covariant and two results about the quenched averaged measure $\mathbb{E} \mathbf{P}_h^{\beta,\omega}$. Translation covariance for the

quenched limit probability and translation invariance of $\mathbb{E}\mathbf{P}_h^{\beta,\omega}$ stem from the same argument, that we give now. Using (5.1) one checks that that for $x \in \mathbb{Z}^d$, we have the following stochastic comparison for the translated measure for finite $N > |x|$:

$$\mathbf{P}_{N-|x|,h}^{\beta,\omega} \leq \mathbf{P}_{N,h}^{\beta,\omega} \Theta_x \leq \mathbf{P}_{N+|x|,h}^{\beta,\omega}, \quad (5.5)$$

where $\mathbf{E}_{N,h}^{\beta,\omega} \Theta_x[h(\phi)] = \mathbf{E}_{N,h}^{\beta,\omega}[h(\Theta_x \phi)]$ for every bounded local continuous function h . Translation covariance of the quenched measure follows by taking $N \rightarrow \infty$, translation invariance for the quenched averaged measure follows by taking the \mathbb{E} expectation of the three terms in (5.5) and by sending $N \rightarrow \infty$.

For the second result on the quenched averaged measure let ∂_h^- and ∂_h^+ denote, respectively, the left and right derivative with respect to h .

Lemma 5.1. *For every h*

$$\mathbb{E}\mathbf{E}_h^{\beta,\omega}[\delta_0] \in [\partial_h^- \mathbf{F}_\infty(\beta, h), \partial_h^+ \mathbf{F}_\infty(\beta, h)]. \quad (5.6)$$

Lemma 5.2. *For every h , if $\{\phi_x\}_{x \in \mathbb{Z}^d}$ is distributed according to $\mathbb{E}\mathbf{P}_h^{\beta,\omega}$, the random field $\{\delta_x\}_{x \in \mathbb{Z}^d}$, is (translation) ergodic.*

Let us see how these two lemmas, and the Markov property, allow to conclude the proof. First of all ergodicity implies that

$$\mathbb{E}\mathbf{P}_h^{\beta,\omega} \left(\text{there exists } x \in \mathbb{Z}^d : \delta_x = 1 \right) \in \{0, 1\}. \quad (5.7)$$

Of course $\mathbb{E}\mathbf{E}_h^{\beta,\omega}[\delta_0]$ is either zero or positive: Lemma 5.1 ensures that this dichotomy precisely corresponds to the localization transition, that is to $h \leq 0$ and $h > 0$ by Theorem 2.2. It also corresponds to the dichotomy (5.7) by elementary arguments.

Consider first the case $h > 0$, that is in the case in which the probability in (5.7) is equal to one, and therefore

$$\mathbf{P}_h^{\beta,\omega} \left(\text{there exists } x \in \mathbb{Z}^d : \delta_x = 1 \right) = 1, \quad \mathbb{P}(d\omega) - \text{a.s.} \quad (5.8)$$

We claim that, $\mathbb{P}(d\omega)$ -a.s., $\mathbf{P}_h^{\beta,\omega}(\phi_y = \infty) = 0$ for every y , hence that $\mathbf{P}_h^{\beta,\omega}(\text{there exists } y \text{ such that } \phi_y = \infty) = 0$. In fact, reasoning by absurd, if there exists y such that $\mathbf{P}_h^{\beta,\omega}(\phi_y = \infty) > 0$ then, by the Markov property (5.3), for every $x \neq y$ we have $\mathbf{P}_h^{\beta,\omega}(\phi_x = \infty, \phi_y = \infty) = \mathbf{P}_h^{\beta,\omega}(\phi_y = \infty) > 0$, so, iterating countably many times the argument, we see that, $\mathbb{P}(d\omega)$ -a.s., $\mathbf{P}_h^{\beta,\omega}(\phi_y = \infty \text{ for every } y \in \mathbb{Z}^d) > 0$, which contradicts the statement (5.8). Therefore the claim is proven and therefore we have also that $\mathbb{E}\mathbf{P}_h^{\beta,\omega}(\text{there exists } y \text{ such that } \phi_y = \infty) = 0$.

On the other hand if $h \leq 0$ we are in the case in which the probability in (5.7) is equal to zero. Hence

$$\mathbf{P}_h^{\beta,\omega} \left(\text{there exists } x \in \mathbb{Z}^d : \delta_x = 1 \right) = 0 \quad \mathbb{P}(d\omega) - \text{a.s.} \quad (5.9)$$

In particular for the same ω 's for any x we have that $\mathbf{P}_h^{\beta,\omega}(\delta_x = 1) = 0$. By the Markov property (5.3) this implies that $\phi_y = \infty$ for at least a $y \sim x$. But we have just seen from the previous argument that $\mathbf{P}_h^{\beta,\omega}(\phi_x = \infty \text{ for every } x) = \mathbf{P}_h^{\beta,\omega}(\phi_y = \infty)$, which in this case is one.

The proof of Theorem 2.3 is complete. \square

Proof of Lemma 5.1 Set $m = \mathbb{E}\mathbf{E}_h^{\beta,\omega}[\delta_0]$. We recall that $\mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega}[\delta_0]$ decreases as N grows. The limit is m by convergence in law, cf. (5.2), because the discontinuity point of δ_0 is $\phi_0 = 1$ and $\mathbf{P}_{h,N}^{\beta,\omega}(\phi_0 = 1) = 0$ as one can see by conditioning on $\mathcal{F}_{\{0\}^c}$ and using (5.3): if the ϕ values on which we condition on the nearest neighbors of 0 are all finite then the conditional measure has a density, otherwise the field at the origin takes the value ∞ . Either ways this conditional probability is zero and the claim follows. By exploiting further the monotonicity under set inclusion of the measure we directly see that for every $\varepsilon > 0$ we can find N_0 such that for $N > N_0$

$$\mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega}[\delta_x] \in [m, m + \varepsilon], \quad (5.10)$$

for every $x \in \tilde{\Lambda}_{N-N_0}$. But then

$$\partial_h \mathbb{E} \log Z_{N,h,\infty}^{\beta,\omega} = \mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega} \left[\sum_{x \in \tilde{\Lambda}_N} \delta_x \right] \leq |\tilde{\Lambda}_{N-N_0}| (m + \varepsilon) + |\tilde{\Lambda}_N \setminus \tilde{\Lambda}_{N-N_0}|, \quad (5.11)$$

and therefore the superior limit as $N \rightarrow \infty$ of the left-hand side, normalized by $|\tilde{\Lambda}_N|$, it is not larger than $m + \varepsilon$. Similarly

$$\partial_h \mathbb{E} \log Z_{N,h,\infty}^{\beta,\omega} = \mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega} \left[\sum_{x \in \tilde{\Lambda}_N} \delta_x \right] \geq |\tilde{\Lambda}_{N-N_0}| m, \quad (5.12)$$

and the inferior limit of this left hand-side, normalized by $|\tilde{\Lambda}_N|$, is not smaller than m . \square

Proof of Lemma 5.2 Let A be a translation invariant event in the σ -algebra generated by $\{\delta_x\}_{x \in \mathbb{Z}^d}$. We can approximate the event by A_M which just depends on $\{\delta_x\}_{x \in \tilde{\Lambda}_M}$ in a way that

$$\mathbb{E}\mathbf{P}_h^{\beta,\omega}(A \Delta A_M) \leq \varepsilon. \quad (5.13)$$

Furthermore we can choose $N > M$ so large that

$$\mathbb{E}\mathbf{E}_{N,h}^{\beta,\omega} \left[\sum_{x \in \tilde{\Lambda}_M} \delta_x \right] - \mathbb{E}\mathbf{E}_h^{\beta,\omega} \left[\sum_{x \in \tilde{\Lambda}_M} \delta_x \right] \leq \varepsilon. \quad (5.14)$$

This is a consequence of the convergence of the sequence of measures, cf. (5.2), because $\mathbb{E}\mathbf{P}_h^{\beta,\omega}(\cup_{x \in \tilde{\Lambda}_M} \{\phi_x = 1\}) = 0$ as can be seen by a conditioning argument like in the very beginning of the proof of Lemma 5.1.

Now set v_N be a vector with all entries 0 except one that is equal to $3N$.

$$\Gamma(N) := \Lambda_N \cup \Theta_{v_N} \Lambda_N. \quad (5.15)$$

Note that Γ_N is composed of two disjoint boxes and thus that under $\mathbf{P}_{\Gamma(N),h}^{\beta,\omega}$, $\{\phi_x\}_{x \in \Lambda_N}$ and $\{\phi_x\}_{x \in \Theta_{v_N} \Lambda_N}$ are independent, so

$$\begin{aligned} \mathbb{E}\mathbf{P}_{\Gamma(N),h}^{\beta,\omega}[A_M \cap \Theta_{v_N} A_M] &= \mathbb{E} \left[\mathbf{P}_{\Lambda_N,h}^{\beta,\omega}(A_M) \mathbf{P}_{\Theta_{v_N} \Lambda_N,h}^{\beta,\omega}(\Theta_{v_N} A_M) \right] \\ &= \mathbb{E} \left[\mathbf{P}_{\Lambda_N,h}^{\beta,\omega}[A_M] \right] \mathbb{E} \left[\mathbf{P}_{\Theta_{v_N} \Lambda_N,h}^{\beta,\omega}(\Theta_{v_N} A_M) \right] = \left(\mathbb{E} \left[\mathbf{P}_{\Lambda_N,h}^{\beta,\omega}[A_M] \right] \right)^2. \end{aligned} \quad (5.16)$$

where in the first equality we used the Markov property and for the second we used independence of the environment in the two boxes.

We assume $N > 2M$ so that $\tilde{\Lambda}_M \cup \Theta_{v_N} \tilde{\Lambda}_M \subset \Gamma(N)$ and the distance of both Λ_M and $\Theta_{v_N} \tilde{\Lambda}_M$ to the boundary of $\Gamma(N)$ is more than $N/2$. We have

$$\mathbb{E} \mathbf{P}_{\Gamma(N),h}^{\beta,\omega} \left[\sum_{x \in \tilde{\Lambda}_M \cup \Theta_{v_N} \tilde{\Lambda}_M} \delta_x \right] - \mathbb{E} \mathbf{P}_h^{\beta,\omega} \left[\sum_{x \in \tilde{\Lambda}_M \cup \Theta_{v_N} \tilde{\Lambda}_M} \delta_x \right] \leq 2\varepsilon, \quad (5.17)$$

as can be directly extracted from (5.14) because the first addendum in the left-hand side can be written as the sum of two terms on which we can apply (5.14) after using translation invariance. Now by stochastic domination we know that there exists a monotone coupling between the two probabilities. For such a coupling $\{\delta_x^1\}_{x \in \tilde{\Lambda}_M \cup \Theta_{v_N} \tilde{\Lambda}_M}$ and $\{\delta_x^2\}_{x \in \tilde{\Lambda}_M \cup \Theta_{v_N} \tilde{\Lambda}_M}$ coincide with probability at least 2ε (we use Markov's inequality together with (5.17)). As a consequence we have

$$\left| \mathbb{E} \mathbf{P}_{\Gamma(N),h}^{\beta,\omega} (A_M \cap \Theta_{v_N} A_M) - \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A_M \cap \Theta_{v_N} A_M) \right| \leq 2\varepsilon. \quad (5.18)$$

Note that in the same manner as for (5.14) – the boundary of A_M is a subset of $\cup_{x \in \tilde{\Lambda}_M} \{\phi_x = 1\}$ – we have also that for M sufficiently large

$$\left| \mathbb{E} \mathbf{P}_{N,h}^{\beta,\omega} (A_M) - \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A_M) \right| \leq \varepsilon. \quad (5.19)$$

By putting everything together (using the triangle inequality and (5.16)) we obtain

$$\begin{aligned} \left| \mathbb{E} \mathbf{P}^{\beta,\omega} (A) - \mathbb{E} \mathbf{P}^{\beta,\omega} (A)^2 \right| &= \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A \cap \Theta_{v_N} A) - \mathbf{P}^{\beta,\omega} (A)^2 \right| \\ &\leq \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A \cap \Theta_{v_N} A) - \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A_M \cap \Theta_{v_N} A_M) \right| \\ &\quad + \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A_M \cap \Theta_{v_N} A_M) - \mathbb{E} \mathbf{P}_{\Gamma(N),h}^{\beta,\omega} (A_M \cap \Theta_{v_N} A_M) \right| \\ &\quad + \left| \mathbb{E} \mathbf{P}_{N,h}^{\beta,\omega} (A_M)^2 - \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A_M)^2 \right| + \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A_M)^2 - \mathbb{E} \mathbf{P}_h^{\beta,\omega} (A)^2 \right| \leq 8\varepsilon. \end{aligned} \quad (5.20)$$

The last inequality comes from the fact that all four terms are smaller than 2ε , the first from (5.13) and translation invariance, the second from (5.18), the third from (5.19) and the last one from (5.13). Since $\varepsilon > 0$ is arbitrary we obtain that $\mathbb{E} \mathbf{P}^{\beta,\omega} (A) \in \{0, 1\}$. \square

Acknowledgements: H. L. acknowledges the support of a productivity grant from CNPq.

APPENDIX A. FREE ENERGY: EXISTENCE AND OTHER ESTIMATES

Theorem A.1. *For every $K \in (-\infty, \infty]$, every $\beta \in I_{\mathbb{P}}$ and every $h \in \mathbb{R}$ we have that the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} \log Z_{N,h,K}^{\beta,\omega,0}. \quad (A.1)$$

exists $\mathbb{P}(d\omega)$ -a.s. and in $L^1(\mathbb{P})$ and the limit is not random.

Therefore (2.7) provides a definition of $F_K(\beta, h)$.

Proof. As long as K is finite the arguments in [6] go through and they yield the result. For $K = \infty$ we observe that, since the partition function decreases as K increases for every K (so, in particular, $F_{\infty}(\beta, h) = \lim_{K \rightarrow \infty} F_K(\beta, h) \geq 0$)

$$\limsup_{N \rightarrow \infty} \frac{1}{N^d} \log Z_{N,h,\infty}^{\beta,\omega,0} \leq F_K(\beta, h), \quad (A.2)$$

$\mathbb{P}(\mathrm{d}\omega)$ -a.s.. On the other hand Lemma A.3 ensures that

$$\liminf_{N \rightarrow \infty} \frac{1}{N^d} \log Z_{N,h,\infty}^{\beta,\omega,0} \geq \mathbb{F}_K(\beta, h) - r(K), \quad (\text{A.3})$$

with $\lim_{K \rightarrow \infty} r(K) = 0$ and this gives (A.1) in the $\mathbb{P}(\mathrm{d}\omega)$ -a.s. sense. The $L^1(\mathbb{P})$ limit can then be obtained by an application of the Dominated Convergence Theorem. \square

As an important technical tool we have the following analog of [11, Prop. 4.2]: the proof is a direct generalization because the potential terms are bounded.

Proposition A.2. *For any value of $u \in \mathbb{R}$, $K \in \mathbb{R}$, $h \in \mathbb{R}$ and $\beta \in I_{\mathbb{P}}$ we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} \mathbb{E} \hat{\mathbf{E}}^u \left[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}} \right] = \mathbb{F}_K(\beta, h). \quad (\text{A.4})$$

Moreover for any u and N one has

$$\frac{1}{N^d} \mathbb{E} \hat{\mathbf{E}}^u \left[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}} \right] \leq \mathbb{F}_K(\beta, h). \quad (\text{A.5})$$

Lemma A.3. *For every K , h and $\beta \in I_{\mathbb{P}}$ we have that the bound*

$$\liminf_{N \rightarrow \infty} \frac{1}{N^d} \log Z_{N,h,\infty}^{\beta,\omega,0} \geq \mathbb{F}_K(\beta, h) - \mathbb{E} [\log (1 + \exp (-K + (\beta\omega_1 - \lambda(\beta) + h)_-))] , \quad (\text{A.6})$$

holds $\mathbb{P}(\mathrm{d}\omega)$ -a.s.. Moreover (A.6) still holds if $\log Z_{N,h,\infty}^{\beta,\omega,0}$ in the left-hand side is replaced by $\mathbb{E} \log Z_{N,h,\infty}^{\beta,\omega,0}$.

Of course $\mathbb{E} [\log (1 + \exp (-K + (\beta\omega_1 - \lambda(\beta) + h)_-))] = o(1)$ as $K \rightarrow \infty$ by the Dominated Convergence Theorem, but the estimate is quantitative. In fact for every $\beta \geq 0$ and $h \geq 0$ we can find $c = c_\beta > 0$ such that for every K sufficiently large we have

$$\mathbb{E} [\log (1 + \exp (-K + (\beta\omega_1 - \lambda(\beta) + h)_-))] \leq \exp(-c_\beta K). \quad (\text{A.7})$$

In fact, it is immediate to see that $c_0 = 1$. For $\beta > 0$ it is sufficient to argue for $h = 0$ and with $Y = \beta\omega_1 - \lambda(\beta)$ we have

$$\begin{aligned} \mathbb{E} [\log (1 + \exp (-K + (Y)_-))] &\leq \\ &\log (1 + \exp (-K/2)) + \mathbb{E} [\log (1 + \exp (-K + (Y)_-)) ; Y < -K/2] \\ &\leq \log (1 + \exp (-K/2)) + \mathbb{E} [(\log 2 + (Y)_-) ; Y < -K/2] , \end{aligned} \quad (\text{A.8})$$

and observe that, since by assumption there exists $a > 0$ such that $\lambda(-a\beta) < \infty$

$$\mathbb{P}(Y < -K/2) = \mathbb{P}\left(-a\beta\omega_1 > \frac{a}{2}K - a\lambda(\beta)\right) \leq \exp\left(\lambda(-a\beta) + a\lambda(\beta) - \frac{a}{2}K\right). \quad (\text{A.9})$$

The conclusion, that is (A.7), is now obtained by applying the Cauchy-Schwarz inequality to the very last term in (A.8).

Proof. We start with observing that the left-hand side in (A.6) is $\mathbb{P}(\mathrm{d}\omega)$ -a.s. equal to

$$\mathbb{F}_K(\beta, h) + \liminf_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbf{P}_{N,h,K}^{\beta,\omega,0} \left(\phi_x \geq 0 \text{ for every } x \in \mathring{\Lambda}_N \right), \quad (\text{A.10})$$

and we have to bound from below the inferior limit in the last expression. For this we observe that if we set $E_A^- := \{\phi \in \mathbb{R}^{\mathring{\Lambda}_N} : \phi_x < 0 \text{ for } x \in A \text{ and } \phi_x \geq 0 \text{ for every } x \in \mathring{\Lambda}_N \setminus A\}$, we have (with the concise notation $Y_x := \beta\omega_x - \lambda(\beta) + h$)

$$\mathbf{P}_{N,h,K}^{\beta,\omega,0}(E_A^-) = \exp(-K|A|) \int_{E_A^-} \frac{\exp\left(\sum_{x \in \tilde{\Lambda}_N} Y_x \delta_x\right)}{Z_{N,h,K}^{\beta,\omega,0}} \mathbf{P}_N^0(d\phi), \quad (\text{A.11})$$

and by performing the change of variables $\tilde{\phi}_x = -\phi_x$ if $x \in A$ and $\tilde{\phi}_x = \phi_x$ otherwise, we see that

$$\int_{E_A^-} \exp\left(\sum_{x \in \tilde{\Lambda}_N} Y_x \delta_x\right) \mathbf{P}_N^0(d\phi) \leq \exp\left(\sum_{x \in A} Y_x\right) \int_{E_\emptyset^-} \exp\left(\sum_{x \in \tilde{\Lambda}_N} Y_x \delta_x\right) \mathbf{P}_N^0(d\phi), \quad (\text{A.12})$$

because such a transformation has (absolute value) of the Jacobian determinant equal to one, $\sum_{x,y} (\tilde{\phi}_x - \tilde{\phi}_y)^2 \leq \sum_{x,y} (\phi_x - \phi_y)^2$, where the sums are over the nearest neighbor (x,y) in $\Lambda_N^2 \setminus (\partial\Lambda_N)^2$, and $\sum_{x \in \tilde{\Lambda}_N} Y_x \delta_x$ under such transformation can decrease of at most $\sum_{x \in A} (Y_x)_-$. Since of course $E_\emptyset^- = \{\phi : \phi_x \geq 0 \text{ for every } x \in \mathring{\Lambda}_N\}$ and since $\sum_{A \subset \mathring{\Lambda}_N} \mathbf{P}_{N,h,K}^{\beta,\omega,0}(E_A^-) = 1$ we see that

$$1 \leq \left(\sum_{A \subset \mathring{\Lambda}_N} \prod_{x \in A} \exp(-K + (Y_x)_-) \right) \mathbf{P}_{N,h,K}^{\beta,\omega,0}(\phi_x \geq 0 \text{ for every } x \in \mathring{\Lambda}_N), \quad (\text{A.13})$$

and since the sum is equal to $\prod_{x \in \mathring{\Lambda}_N} (1 + \exp(-K + (Y_x)_-))$, the claim in Lemma A.3 follows by applying the law of large numbers to the family of L^1 IID random variables $\{\log(1 + \exp(-K + (Y_x)_-))\}_{x \in \mathbb{Z}^d}$. \square

REFERENCES

- [1] K. S. Alexander and N. Zygouras, *Equality of critical points for polymer depinning transitions with loop exponent one*, Ann. Appl. Probab. **20** (2010), 356-366.
- [2] E. Bolthausen, J.-D. Deuschel and O. Zeitouni, *Entropic repulsion of the lattice free field*, Commun. Math. Phys. **170** (1995), 417-443.
- [3] E. Bolthausen, J.-D. Deuschel and O. Zeitouni, *Absence of a wetting transition for a pinned harmonic crystal in dimensions three and larger*, J. Math. Phys. **41** (2000), 1211-1223.
- [4] P. Caputo and Y. Velenik, *A note on wetting transition for gradient field*, Stoch. Proc. Appl. **87** (2000), 107-113.
- [5] A. Chiarini, A. Cipriani and R. S. Hazra, *A note on the extremal process of the supercritical Gaussian Free Field*, Electron. Comm. Probab. **20** (2015) 74.
- [6] L. Coquille and P. Milos, *A note on the discrete Gaussian free field with disordered pinning on \mathbb{Z}^d , $d \geq 2$* , Stoch. Proc. and Appl. **123** (2013) 3542-3559.
- [7] J.-D. Deuschel, *Entropic repulsion of the lattice free field. II. The 0-boundary case*, Commun. Math. Phys. **181** (1996), 647-665.
- [8] J.-D. Deuschel and G. Giacomin, *Entropic repulsion for the free field: pathwise behavior in $d \geq 3$* , Comm. Math. Phys. **206** (1999), 447-462.
- [9] G. Giacomin, *Random polymer models*, Imperial College Press, World Scientific (2007).
- [10] G. Giacomin, *Disorder and critical phenomena through basic probability models*, École d'été de probabilités de Saint-Flour XL-2010, Lecture Notes in Mathematics **2025**, Springer, 2011.
- [11] G. Giacomin and H. Lacoïn, *Pinning and disorder relevance for the lattice Gaussian free field*, arXiv:1501.07909, to appear on JEMS
- [12] A. B. Harris, *Effect of random defects on the critical behaviour of Ising models* J. Phys. C **7** (1974), 1671-1692.

- [13] H. Lacoin, *Pinning and disorder for the Gaussian free field II: the two dimensional case*, arXiv:1512.05240 [math-ph].
- [14] J. L. Lebowitz and C. Maes, *The effect of an external field on an interface, entropic repulsion*, J. Statist. Phys. **46** (1987), 39-49.
- [15] J. Sohler, *The scaling limits of the non critical strip wetting model*, Stoch. Proc. Appl. **125** (2015), 3075-3103.
- [16] Y. Velenik, *Localization and delocalization of random interfaces*, Probab. Surv. **3** (2006), 112-169.
- [17] O. Zeitouni, *Branching random walks and Gaussian fields*, lecture notes, available on the webpage of the author

UNIVERSITÉ PARIS DIDEROT, SORBONNE PARIS CITÉ, LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, UMR 7599, F- 75205 PARIS, FRANCE.

IMPA, INSTITUTO DE MATEMÁTICA PURA E APLICADA, ESTRADA DONA CASTORINA 110 RIO DE JANEIRO, CEP-22460-320, BRASIL.